B.Sc. Semester-VI Examination, 2022-23

MATHEMATICS [Honours]

Course ID : 62116 **Course Code : SH/MTH/603/DSE-3**

Course Title : Number Theory

Time : 2 Hours Full Marks : 40

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meaning.

1. Answer any **five** of the following questions: $2\times5=10$

a) If gcd(a, b) = 1 and $c|a$, then show that gcd(c, b) = 1.

b) Show that $\phi(3n)=3\phi(n)$ if $3|n$.

c) Find the least positive residue of 2^{44} (mod 89).

d) Prove that square of the any integer is of the form $5k\pm1$.

e) Find $7^{-1}(\text{mod}\,20)$.

f) Find the general solution of $12x - 17y = -7$.

g) Show that $a^{21}\equiv a(\text{mod}\ 15)$ for all a.

h) Verify that 1000! terminates in 249 zeros.

[Turn Over]

2. Answer any **four** from the following questions: $5\times4=20$

a) i) Find the values of $n\geq1$ for which $1!+2!+3!+...+n!$ is a perfect square.

ii) Prove that the integer $111^{333}+333^{111}$ is divisible by 7. 3+2

b) i) If the integer a has order k modulo n and $h>0$, then a^h has order $k/\gcd(h, k)$ modulo n.

ii) Prove that $\phi(2^n-1)$ is a multiple of n for any $n>1$. 3+2

c) i) Show that every number and its cube when divided by 6 leave the same remainder.

ii) Show that $a^{4b+1}-a$ is divisible by 30. 3+2

d) i) If n is a composite number, then prove that $(n, \phi(n))>1$.

ii) If $n=p_1^{a_1}p_2^{a_2}...p_k^{a_k}\ (n>1)$, where p_i are prime numbers and $a_i\in\mathbb{Z}^+$, then prove that $\tau(n)=(a_1+1)(a_2+1)...(a_k+1)$.

e) State Chinese remainder theorem, and hence solve the system of linear congruences:

$$x\equiv2(mod\,3),\ x\equiv3(mod\,5),\ x\equiv4(mod\,7).$$

f) i) If $\gcd(a, b) = 1$, then show that $\gcd\left(a+b, a^2-ab+b^2\right)=1$ or 3.

ii) Prove that the product of any three consecutive integers is divisible by 6.

3. Answer any **one** of the following questions:

$10\times1=10$

a) i) Show that $a^{18} - b^{18}$ is divisible by 133 if a and b both are prime to 133.

ii) If p is a prime, then show that $2(p-3)! \equiv -1 (mod\ p)$.

iii) Using the theory of congruence, find the remainder if $1 + 2^5 + 3^5 + ... + 100^5$ is divisible by 4. 3+3+4

b) i) Use Fermat's theorem to prove that, if p is an odd prime, then

$$1^{p-1}+2^{p-1}+3^{p-1}+...+(p-1)^{p-1} \equiv -1(\text{mod } p).$$

ii) If $n = p_1^{k_1} p_2^{k_2} ... p_r^{k_r}$ is the prime factorization of $n>1$ then

$$1>\frac{n}{\sigma(n)}>\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)...\left(1-\frac{1}{p_r}\right).$$

iii) Determine the highest power of 3 dividing 80!. 4+3+3=10

B.Sc. Semester-VI Examination, 2022-23

MATHEMATICS [Honours]

Course ID : 62116 **Course Code : SH/MTH/603/DSE-3**

Course Title : Mechanics

Time : 2 Hours Full Marks : 40

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meaning.

1. Answer any **five** of the following questions:

$2\times5=10$

a) Write down the expressions of potential and kinetic energies of a simple pendulum of length l oscillating in a uniform gravitational field.

b) If a point moves so that its angular velocity about two fixed points is the same, prove that it describes a circle.

c) Two forces $2p$ and p act along the lines whose equations are $y = x\tan\alpha$, $z = c$ and $y = -x\tan\alpha$, $z = -c$ respectively. Find the equation of the central axis.

d) If two particles P and Q describe the same ellipse under the same central force to the centre C, prove that the area of the triangle CPQ is invariable.

e) Prove that at an apse the particle is moving at right angles to the radius vector.

f) Find the relation between the magnitude of angular velocity and linear velocity.

g) Find the condition of a stable motion when a particle describes a path which is nearly a circle of radius a about a centre of force $f(r)$ at its centre.

h) A shell of mass m is ejected from a gun of mass M by an explosion which generates kinetic energy E. Prove that the initial velocity of the shell is

$$\sqrt{\frac{2ME}{(M+m)m}}.$$

2. Answer any **four** of the following questions: 5×4=20

a) Six equal heavy beams are freely jointed at their ends to form a hexagon and are placed in a vertical plane with one beam resting on a horizontal plane; the middle points of the two upper slant beams, which are inclined at an angle θ to the horizon, are connected by a light rod. Show that its tension is $6W\cot\theta$, where W is the weight of each beam.

b) Find the centre of gravity of the surface formed by the revolution of the cardioid $r = a(1+\cos\theta)$ about its axis.

c) A particle of mass m falls from rest under gravity in a medium whose resistance is $mk(velocity)^2$, show that at any time t if v be the velocity and x be the distance descended, then

$$v = c\tanh\frac{gt}{c}, \quad x = \frac{c^2}{g}\log_e\left(\cosh\left(\frac{gt}{c}\right)\right)$$

where c is the terminal velocity and g is the acceleration due to gravity. 5

d) A uniform chain, of length l, is to be suspended from two points, A and B in the same horizontal line so that either terminal tension is n times that at the lowest point. So that the span AB must be

$$\frac{l}{n^2-1}\log_e\left(n+\sqrt{n^2-1}\right).$$

e) A shell of mass m is fired from a gun of mass M which can recoil freely on a horizontal base, and the elevation of the gun is α. Prove that the initial inclination of the path of the shell to the horizon is $\tan^{-1}\left\{\left(1+\frac{m}{M}\right)\tan\alpha\right\}$. 5

f) A car of mass m starts from rest and moves on a level road under a constant frictional resistance, the engine working at a constant rate P. If the maximum speed be V and the speed u be attained after travelling a distance s in time t, then show that $t = \frac{s}{V} + \frac{mu^2}{2P}$.

3. Answer any **one** of the following questions: 10×1=10

a) i) A system of coplanar forces is acting on a rigid body. Using the principle of virtual work, deduce the condition of equilibrium of the system.

ii) A particle is projected vertically upwards from the surface of earth with a velocity just sufficient to carry it to infinity. Show that the time it takes in reaching a height h is

$$\frac{1}{3}\sqrt{\frac{2a}{g}}\left\{\left(1+\frac{h}{a}\right)^{\frac{3}{2}}-1\right\},$$ where a is the radius of the earth and g is the value of gravity at its surface. 5+5=10

b) i) A uniform chain of length $2a$ is hang over a smooth peg so that the length of it on two sides are $a + b$ and $a - b$. If motion starts at this point of time, find the time when the chain leaves the peg.

ii) A particle moves in a smooth tube in the form of a catenary, being attracted to the directrix by a force proportional to the distance from it. Show that the motion is simple harmonic. 5+5=10

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